AIChE

The Number "*e*": Concept, Definition, Calculation, and Accurate Formula

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*This article demonstrates a new method to determine the digits for the value of "*e*" Napier's constant or Euler's number. The estimate works out to a single equation. In process safety, several failure probability models use values of "*e*" to a power. Thus, novel methods to estimate the value of "*e*" may be of interest to a process safety engineer.* © 2019 American Institute of Chemical Engineers Process Saf Prog 38: e12052, 2019
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INTRODUCTION

A few years ago, an analysis I conducted of failure probability curves $F(t) = 1 - e^{-\lambda t}$ in equipment devices used for the protection of Hazardous Industrial Processes triggered my curiosity for, and interest in, unraveling the calculation of the value of the famous irrational number "***e***" (Napier's constant in 1614, Jacob Bernoulli binomial in 1683 and Euler's number in 1727), one of the most important numbers in various branches of mathematics, extremely useful for practical purposes and the base of natural logarithms.

Here is how I defined and calculated it.

CONCEPT, DEFINITION, CALCULATION, AND ACCURATE FORMULA

With respect to Figure 1, I asked myself the following question: which is the value of number "*a*" (which I subsequently identified as "*e*") where the tangent line, at any of the points of function "a^x", always has a slope equal to the value of "a^x" in the respective point "*x*"?

As Figure 1 shows, this means that tangent line AB at any point B of function "a^x" makes—with the "*x*" axis- the angle BAC, which will always have a base AC = 1 to meet the condition of slope "*p*" being p = BC/AC=BC = "a^x" for any "*x*".

Considering Figure 2, we can see that:

- The value of the function at x_1 (point R) is $a^{x1} = a^{x + \Delta x}$
- The value of the function at x_2 (point S) is $a^{x2} = a^{x - \Delta x}$
- Therefore, the value of cathetus RT is = $a^{x + \Delta x} - a^{x - \Delta x}$
- The value of cathetus ST is = $x_1 - x_2 = 2\Delta x$
- And the value of slope "*p*" of secant line RS is p = RT/ST.

The aim was to ensure that, when secant line RS becomes tangent at point B (whereby both Δx will tend to zero and points R and S will meet with point B), the slope—as a condition—always be.

p = RT/ST = a^x, for any "*x*".

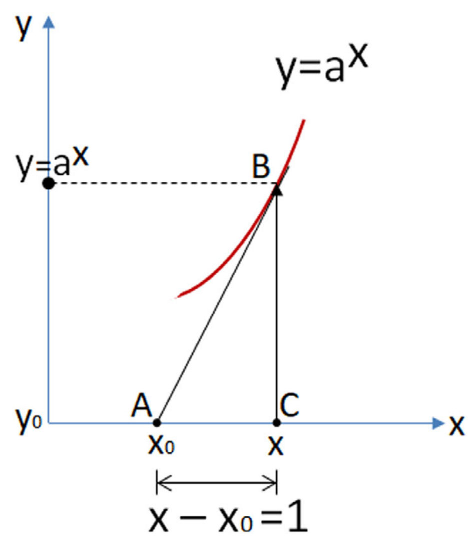

Figure 1. Tangent line AB with AC = 1. [Color figure can be viewed at **wileyonlinelibrary.com**]

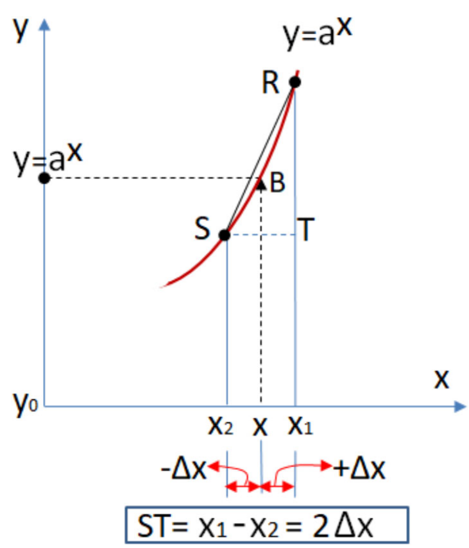

Figure 2. Secant line RS. [Color figure can be viewed at **wileyonlinelibrary.com**]

Upon rearranging, we get:

$$p = a^x = \mathrm{RT/ST} = \left(a^{x+\Delta x} - a^{x-\Delta x}\right)/2\Delta x$$
$$= \left(a^x * a^{+\Delta x} - a^x * a^{-\Delta x}\right)/2\Delta x = a^x$$

Upon dividing both terms by a^x, rearranging the formula, and multiplying everything by $a^{+\Delta x}$, we obtain the following quadratic equation:
$(a^{+\Delta x})^2 - 2\Delta x^* \, a^{+\Delta x} - 1 = 0$, the solution being

$$a^{+\Delta x} = 1/2\left(2\Delta x + \sqrt{(2\Delta x)^2 + \mathbf{4}}\right) = \Delta x + \sqrt{(\Delta x)^2 + \mathbf{1}}$$

By raising both terms to $1/\Delta x$, we get:

$$\left(\mathrm{a}^{+\Delta x}\right)^{1/\Delta x} = a = \boldsymbol{e} = \left(\Delta x + \sqrt{(\Delta x)^2 + 1}\right)^{1/\Delta x}$$

For the sake of simplicity, I used $\Delta x = 1/n$ and, upon rearranging, we get:

$$\boldsymbol{e} = \left(\sqrt{1 + (1/n)^2} + 1/n\right)^n$$

That should be stated as $e = \lim_{n \to \infty} \left(\sqrt{1 + (1/n)^2} + 1/n\right)^n$

This is the accurate formula of the number that I set out to find, the so-called "***e***", the one that meets the condition that the slope of tangent line at e^x, for any "x", is always equal to the value of function e^x.

It should be noted that, when we reduce ΔX significantly by significantly increasing the value of "n", the $(1/n)^2$ becomes almost negligible, whereupon the equation of the popular binomial $(1 + 1/n)^n$ begins to resemble the equation developed here, thus confirming that only when "n" grows to the infinity ($n \to \infty$) does the binominal begin to approach the known value of e = 2.71828182845904523...

*I will now compare the results of both formulas calculating with n = 10^7, (that is, $\Delta x = 1/10^7$):

*The known value of "***e***" with $n \to \infty$ (infinity) is ***e* = 2.71828182845904523**...

*With my new formula and with only $n = 10^7$, it is ***e* = 2.71828182845904**07…

which quickly tends toward "***e***", here coinciding up to decimal "14".

*And the binomial $(1 + 1/n)^n$ with $n = 10^7$ barely results in ***e* = 2.718281**6925449662 coinciding only up to the 6th decimal.

A comparative table of the values resulting from both equations for different "n" values is included below.

CONCLUSION

This comparative table confirms the proposition that the value of "***e***", resulting from the formula I developed here, is "the one meeting the condition that the slope of any line tangent to curve $\boldsymbol{e}^x$ should have—for any "x"—a value equal to that of function $\boldsymbol{e}^x$ at the respective point x", while the binominal formula $(1 + 1/n)^n$ only approaches the value "e" when "n" moves toward infinity "$n \to \infty$".

n	$1/n$	$(1/n)^2$	$(1 + 1/n)^n$	$\boldsymbol{e} = \left[\sqrt{1+(1/n)^2} + 1/n\right]^n$
1	1/1	$(1/1)^2$	**2**.000000..	**2**.4142135623...
10	1/10	$(1/10)^2$	**2**.593742..	**2.71**37753649...
100	1/100	$(1/100)^2$	**2.7**04813..	**2.7182**365261...
1,000	1/1,000	$(1/1{,}000)^2$	**2.71**6923..	**2.718281**3754...
10,000	1/10,000	$(1/10{,}000)^2$	**2.718**145..	**2.71828182**3928..
100,000	1/100,000	$(1/100{,}000)^2$	**2.7182**68..	**2.7182818284**137405..
10^7	$1/10^7$	$(1/10^7)^2$	**2.718281**692.	**2.71828182845904**07..
10^{12}	$1/10^{12}$	$(1/10^{12})^2$	**2.71828182845**7686...	**2.718281828459045235**...
				Known value ***e* = 2.718281828459045235360...**